§2.6

Monoidal functors between cat. of graded v.s

Let G_1, G_2 be groups, A – abelian group,

let $\omega_i \in Z^3(G_i, A)$ $i = 1, 2$

($G_1, G_2 \curvearrowright A$ trivial)

Let $\mathcal{C}_i = \mathcal{C}_{G_i}^{\omega_i}$ $i = 1, 2$ example 2.3.8

recall: $\mathcal{C}_G^{\omega} = \mathcal{C}_G^{\omega}(A)$

obj: δ_g, $g \in G$

morphism: $\mathrm{Hom}(\delta_g, \delta_g) = A$

ϕ otherwise

$\otimes$: multiplication

associativity:

$$a^{\omega}_{\delta_g, \delta_h, \delta_m} = \omega(g,h,m)\, \mathrm{id}_{\delta_{ghm}}$$

$$(\delta_g \otimes \delta_h) \otimes \delta_m \to \delta_g \otimes (\delta_h \otimes \delta_m)$$

Any monoidal functor $F: \mathcal{C}_1 \to \mathcal{C}_2$ defines by restriction to simple obj a group hom $f: G_1 \to G_2$

$$F(\delta_g \otimes \delta_h) \overset{J}{\cong} F(\delta_g) \otimes F(\delta_h)$$

$$\begin{tikzcd} F(\delta_g \otimes \delta_h) \arrow[r, phantom, "\overset{J}{\cong}"] \arrow[d, phantom, "\shortparallel"] & F(\delta_g) \otimes F(\delta_h) \arrow[d, phantom, "\shortparallel"] \\ \delta_{\boxed{f(gh)}} & \delta_{f(g)} \otimes \delta_{f(h)} \arrow[d, phantom, "\shortparallel"] \\ & \delta_{\boxed{f(g)f(h)}} \end{tikzcd}$$

Using axiom (2.2.3),

we see that a monoidal structure on F is given by

$$J_{g,h} = \mu(g,h)\, \mathrm{id}_{\delta_{f(gh)}} : F(\delta_g) \otimes F(\delta_h) \xrightarrow{\sim} F(\delta_{gh})$$

$g, h \in G_1$

where $\mu : G_1 \times G_1 \to A$ is a function s.t

$$\underline{\omega_1(g,h,l)}\, \mu(gh,l)\, \mu(g,h) = \mu(g,hl)\, \mu(h,l)\, \omega_2(f(g), f(h), f(l))$$

$\forall\, g, h, l \in G_1$

$$\begin{tikzcd} (F(X) \otimes F(Y)) \otimes F(Z) \arrow[r, "a"] \arrow[d, "J \otimes \mathrm{id}"'] & F(X) \otimes (F(Y) \otimes F(Z)) \arrow[d, "\mathrm{id} \otimes J"] \\ F(X \otimes Y) \otimes F(Z) \arrow[d, "J"'] & F(X) \otimes F(Y \otimes Z) \arrow[d, "J"] \\ F((X \otimes Y) \otimes Z) \arrow[r, "\boxed{F(a)}"] & F(X \otimes (Y \otimes Z)) \end{tikzcd}$$

$X, Y, Z \to \delta_g, \delta_h, \delta_l$

↳ : $\omega_1(g,h,l)\, \mu(gh,l)\, \mu(g,h)$

↴ : - - -.

$$\underline{\omega_1 = f^* \omega_2 \cdot d_3(\mu)}$$

$G \times G \times G \to A$

i.e ω_1 and $f^* \omega_2$ are cohomology in $Z^3(G_1, A)$

Conversely. given a group hom $f: G_1 \to G_2$

any function $\mu: G_1 \times G_1 \to A$ satisfying (2.3.1) gives rise to a monoidal functor $F: \mathcal{C}_1 \to \mathcal{C}_2$ defined by $F(\delta_g) = \delta_{f(g)}$ with monoidal structure given by (2.3.2)

This functor is an equivalence $\Leftrightarrow$ f is an iso.

To summarize, monoidal functor $\mathcal{C}_{G_1}^{\omega_1} \to \mathcal{C}_{G_2}^{\omega_2}$ correspond to pairs (f, μ) where $f: G_1 \to G_2$ hom s.t. ω_1 and $f^*\omega_2$ are cohomologous, and μ is function satisfying (2.3.1)

$$\omega_1 = f^*\omega_2 \cdot d_3(\mu)$$

Let $F_{f,\mu}$ denote the corresponding functor.

Let's determine $F_{f,\mu}(\delta_g) \longrightarrow F_{f',\mu'}(\delta_g)$

Clearly, such a transformation exists

$\Leftrightarrow f = f'$, ...

is always an isomorphism and is determined by a collection of morphisms $\eta_g: \delta_{f(g)} \to \delta_{f(g)}$

$\eta_g \in A$

satisfying the equation

$$\mu'(g,h)(\eta_g \otimes \eta_h) = \eta_{gh}\,\mu(g,h)$$

i.e. $\mu = \mu' \cdot d_2(\mu)$

$$\begin{tikzcd}
F'(X) \otimes F'(Y) \arrow[r, "J'"] \arrow[d, "\eta_X \otimes \eta_Y"'] & F'(X \otimes Y) \arrow[d, "\eta_{X \otimes Y}"] \\
F^2(X) \otimes F^2(Y) \arrow[r, "J^2"] & F^2(X \otimes Y)
\end{tikzcd}$$

$\mu'(g,h)(\eta_g \otimes \eta_h) = \eta_{gh}\, \mu(g,h)$

$d_2 f(g,h) = f(h) - f(gh) + f(g)$

i.e. $\mu = \mu' \cdot d_2(\underset{\eta}{\cancel{\mu}})$

$G_1 \to A$

Conversely, every function $\eta : G_1 \to A$ satisfying (2.33) gives rise to a morphism of monoidal functors $\eta : F_{f,\mu} \to F_{f,\mu'}$

Thus, monoidal functors $F_{f,\mu}$ and $F_{f',\mu'}$ are iso $\Leftrightarrow f = f'$ and μ is cohom- to μ'

Prop 2.6.1

torsor:

X is a torsor over a grp G:

$G \circlearrowright X$ transitive

and the stabilizer subgrp of every point is trivial.

(i) The set of iso between $F_{f,\mu}$ and $F_{f,\mu'}$: $C_{G_1}^{\omega_1} \to C_{G_2}^{\omega_2}$ is a torsor over the grp $H^1(G_1, A) \cong \mathrm{Hom}(G_1, A)$

$\underline{H^1(G,A)} = \boxed{\mathrm{Ker}\, d_2} / \mathrm{Im}\, d_1 = 0$

$\mu = \mu' \cdot d_2(\eta)$

(ii) Fix $f: G_1 \to G_2$

the set of μ parametering iso classes of monoidal functors $F_{f,\mu}$ is a torsor over $H^2(G, A)$

$$H^2(G, A) = \operatorname{Ker} d_3 / \operatorname{Im} d_2$$

$$\underline{\omega_1 = f^* \omega_2 \cdot d_3(\mu)}$$

$$\underline{\mu = \mu' \cdot d_2(y)}$$

(iii) Equivalence classes of monoidal categories C_G^ω are parameterized by the set $\boxed{H^3(G, A) / \operatorname{Out}(G)}$

where $\operatorname{Out}(G)$ is the group of Outer auto— of G.

取 f: $G \xrightarrow[id]{} G$, 或 ω

$$\Rightarrow \underline{C_G^\omega} \underset{(2.3.1)}{\longleftrightarrow} H^3(G, A)$$

$$\underline{\omega_1 = f^* \omega_2 \cdot d_3(\mu)}$$

$f \in \operatorname{Aut}(G)$

$$f \curvearrowright [\omega_2] = [\omega_1]$$

$$C_G^{\omega_1} \underset{f,\mu}{\sim} G_G^{\omega_2}$$

Remk 2.6.2

$A = \mathbb{k}^\times$

replace C_G^ω by Vec_G^ω

~~replace~~ require the monoidal functors we consider are additive

To see this, it's enough to note that:

by def, $\forall$ morphism η of monoidal functors,

$$\eta_{\textcircled{1}} \neq 0$$

So Equation (2.32) with $\underline{(h = g^{-1})}$ implies all $\eta_{\textcircled{g}} \neq 0$.

$F_{f,\mu}$ $F_{f,\mu'}$

Thus, if a morphism $\eta : F_{f,\mu_0} \to F_{f',\mu}$ exists, then it is an isomorphism, and we must have $f = f'$.

$\underline{V\ G\ Vec_G^{\omega}}$ g_V

Remark 2.6.3

It may be assumed without loss of generality that the cocycle ω is normalized i.e. $\omega(g,1,h) = 1$ and $\underline{\text{thus}}$ $l_{\delta_g} = r_{\delta_g} = id_{\delta_g}$

Indeed, we claim that any 3-cocycle ω is cohomologous to a normalized one. $\omega \in Z^3$

let μ be any 2-cochain s.t

$$\mu(g,1) = \underline{\omega(g,1,1)}$$

$$\mu(1,h) = \underline{\omega(1,1,h)^{-1}}$$

3-cocycle ω: for g_1, g_2, g_3, g_4

$$\omega(g_1g_2, g_3, g_4)\,\omega(g_1, g_2, g_3g_4) = \omega(g_1, g_2, g_3)\,\omega(g_1, g_2g_3, g_4)\,\omega(g_2, g_3, g_4)$$

Let $g_1 = g$, $g_2 = 1$, $g_3 = 1$, $g_4 = h$

$$\omega(g,1,h)\,\omega(g,\cancel{1},h) = \omega(g,1,1)\,\omega(g,\cancel{1,h})\,\omega(1,1,h)$$

$\Rightarrow \omega(g,1,h) = \omega(g,1,1)\,\omega(1,1,h)$

$d_3(\mu)(g,1,h)$

$= \omega(1,1,h)^{-1}\,\omega(g,1,1)^{-1}$

$\Rightarrow \underbrace{\boxed{\omega\, d_3(\mu)}}_{\text{normalized}}(g,1,h) = 1$

Example 2.6.4

Let $G = \mathbb{Z}/n\mathbb{Z}$, $n>1$, $k = \mathbb{C}$

Consider the cohomology of $\mathbb{Z}/n\mathbb{Z}$.

Since $H^i(\mathbb{Z}/n\mathbb{Z}, \boxed{\mathbb{C}}) = 0$, $\underline{\forall i>0}$

$\boxed{\mathbb{Z}}$

$\boxed{\mathbb{C}^\times}$

$$0 \to \mathbb{Z} \to \mathbb{C} \to \mathbb{C}/\mathbb{Z} \to 0$$

$\cong \mathbb{C}^\times$

$z \to e^{2\pi i z}$

$$\begin{tikzcd} \cdots \arrow[r, "(g-1)"] & \underset{P_2}{\mathbb{Z}G} \arrow[r, "(1+g+\cdots+g^{n-1})"] & \underset{P_1}{\mathbb{Z}G} \arrow[r, "(g-1)"] & \underset{P_0}{\mathbb{Z}G} \arrow[r] & \mathbb{Z} \end{tikzcd}$$

$H^i(\mathbb{Z}/n\mathbb{Z}, \underline{\mathbb{C}^\times})$

$\underline{H^i(\mathbb{Z}/n\mathbb{Z}, \mathbb{C}^\times) \cong H^{i+1}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z})}$

$$H^i(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}) = \begin{cases} \mathbb{Z} & i=0 \\ 0 & i \text{ odd} \\ \mathbb{Z}/n\mathbb{Z} & i \text{ even} \end{cases}$$

As we saw in Exercise 1.7.5

the graded ring $H^*(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z})$

$$\begin{aligned} &\ \ \Vert \\ &\mathbb{Z}[x]/(nx) = \mathbb{Z} \oplus x(\mathbb{Z}/n\mathbb{Z})[x] \end{aligned}$$

Moreover, as a module over $\mathrm{Aut}(\mathbb{Z}/n\mathbb{Z}) = (\mathbb{Z}/n\mathbb{Z})^\times$

we have $H^2(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}) \simeq H^1(\mathbb{Z}/n\mathbb{Z}, \mathbb{C}^\times) = (\mathbb{Z}/n\mathbb{Z})^\vee$

Therefore, using the graded ring structure we find that

$\mathbb{Z}/n\mathbb{Z}$

$$H^{2m-1}(\mathbb{Z}/n\mathbb{Z}, \mathbb{C}^\times) \simeq H^{2m}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}) \simeq ((\mathbb{Z}/n\mathbb{Z})^\vee)^{\otimes m}$$

as an $\mathrm{Aut}(\mathbb{Z}/n\mathbb{Z})$-module

Let us give an explicit formula for the 3-cocycles on $\mathbb{Z}/n\mathbb{Z}$

$$\phi(i,j,k) = \varepsilon^{\frac{si(j+k-(j+k)')}{n}}$$

ε — primitive nth root

$s \in \mathbb{Z}/n\mathbb{Z}$